A Closed Formula for the Riemann Normal Coordinate Expansion

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Abstract

We derive an integral representation which encodes all coefficients of the Riemann normal coordinate expansion and also a closed formula for those coefficients. In gauge theory, one often uses Fock-Schwinger gauge [1, 2] to achieve manifest covariance in the calculation of effective actions, anomaly densities, or other quantities. Fock-Schwinger gauge "centered at 0" can be defined by the condition

$$y^{\mu}A_{\mu}(y) = 0. \tag{1}$$

Locally in some neighbourhood of 0, this condition can be solved in terms of the following integral representation connecting the gauge potential and the field strength tensor [3, 4],

$$A_{\mu}(y) = y^{\rho} \int_0^1 \mathrm{d}v v F_{\rho\mu}(vy) \tag{2}$$

(see our eqs. (9-13)). More explicitly, in this gauge one can express the coefficients of the Taylor expansion of A at x = 0 in terms of the covariant derivatives of F,

$$A_{\mu}(y) = \sum_{n=0}^{\infty} \frac{(y \cdot \partial_x)^n}{n!(n+2)} y^{\rho} F_{\rho\mu}(0) = \sum_{n=0}^{\infty} \frac{(y \cdot D_x)^n}{n!(n+2)} y^{\rho} F_{\rho\mu}(0).$$
 (3)

Despite of the direct relation between eqs. (2) and (3), the former version turns out to have advantages for certain types of calculations in Yang-Mills theory [4, 5].

The analogue to Fock-Schwinger gauge in gravity is the choice of a Riemann normal coordinate system (see, e. g., [6, 7, 8, 9, 10]). A normal coordinate system on a Riemannian manifold centered at 0 can be defined by

$$g_{\mu\nu}(0) = \delta_{\mu\nu}, \quad y^{\mu}g_{\mu\nu}(y) = y^{\mu}g_{\mu\nu}(0).$$
 (4)

Alternatively, the second condition may be replaced by the following equivalent condition on the affine connection

$$y^{\mu}y^{\nu}\Gamma^{\lambda}_{\mu\nu}(y) = 0. \tag{5}$$

Here $\Gamma^{\lambda}_{\mu\nu}$ denotes the Christoffel symbol for the Levi-Civita connection. Locally this condition determines the coordinate system up to a rigid rotation. The second condition clearly shows the similarity to the Fock-Schwinger gauge eq. (1). It also shows that in those coordinates straight lines running through the origin parametrize geodesics.

In Riemann normal coordinates the Taylor coefficients of the metric tensor at 0 can be expressed in terms of the covariant derivatives of the Riemann curvature tensor. The Taylor expansion starts as follows¹,

$$g_{\mu\nu}(y) = \delta_{\mu\nu} + \frac{1}{3} R_{\mu\alpha\beta\nu}(0) y^{\alpha} y^{\beta} + \frac{1}{6} \nabla_{\gamma} R_{\mu\alpha\beta\nu}(0) y^{\alpha} y^{\beta} y^{\gamma}$$

$$+ \frac{2}{45} R_{\mu\alpha\beta\lambda}(0) R^{\lambda}_{\gamma\delta\nu}(0) y^{\alpha} y^{\beta} y^{\gamma} y^{\delta} + \frac{1}{20} R_{\mu\alpha\beta\nu;\gamma\delta}(0) y^{\alpha} y^{\beta} y^{\gamma} y^{\delta} + O(y^{5})$$
 (6)

(in these coordinates and at the origin there is no need to distinguish between co- and contravariant indices).

$$R^{\alpha}_{\ \beta\gamma\delta} = \Gamma^{\alpha}_{\ \beta\delta,\gamma} - \Gamma^{\alpha}_{\ \beta\gamma,\delta} + \Gamma^{\mu}_{\ \beta\delta}\Gamma^{\alpha}_{\ \mu\gamma} - \Gamma^{\mu}_{\ \beta\gamma}\Gamma^{\alpha}_{\ \mu\delta}, \quad R_{\alpha\beta} = R^{\gamma}_{\ \alpha\gamma\beta}, \quad R = R^{\mu}_{\ \mu}.$$

¹ Our conventions are

Riemann normal coordinates are a standard tool in the proof of differential geometric identities. To cite a prominent example, they play a pivotal role in the local heat equation proof of the Atiyah-Singer index theorem [11]. In physics they are, for example, widely used for nonlinear σ -model calculations in curved spacetime backgrounds [7, 9, 10, 12].

By differentiation of eq. (6) one obtains a similar expansion for $\Gamma^{\lambda}_{\mu\nu}$,

$$\Gamma^{\lambda}_{\mu\nu}(y) = \frac{1}{3} \Big(R_{\nu\alpha\mu}^{\quad \lambda}(0) + R_{\mu\alpha\nu}^{\quad \lambda}(0) \Big) y^{\alpha} + \dots$$
 (7)

In contrast to its gauge theory analogue eq. (3), this expansion contains arbitrary powers of the curvature tensor. As far as is known to the authors, neither a closed formula for its coefficients has been given in the literature, nor an equivalent of the integral representation eq. (2). It is the purpose of the present note to fill this gap.

In the literature one finds various ways of deriving eq. (6) from eqs. (4) or (5), going back at least to 1925 [9, 13, 14, 15]. We will essentially follow [15] in the first part of our argument. First notice that it suffices to find the Taylor expansion of the vielbein $e^a_{\mu}(x)$, since

$$g_{\mu\nu}(x) = e^a_{\mu}(x)e^b_{\nu}(x)\delta_{ab}.$$
 (8)

Finding an expression for the vielbein in terms of the curvature effectively involves a twofold integration. As a first step, we express the vielbein connection ω in terms of the curvature. For this purpose, consider the Lie transport of ω with respect to the radial vector y. Writing the Lie derivative in terms of the interior product i_y and the exterior derivative d,

$$\mathcal{L}_{y} = i_{y} d + di_{y}, \tag{9}$$

one has

$$\mathcal{L}_{y}\omega = i_{y}d\omega + d(i_{y}\omega). \tag{10}$$

We choose synchronous gauge, i.e. Fock-Schwinger gauge for the vielbein connection

$$i_y \omega = 0 \tag{11}$$

This removes the second term. Using the Cartan structure equation $R = d\omega + \omega \wedge \omega$ one obtains

$$\mathcal{L}_y \omega_\mu^{\ a} \mathrm{d}y^\mu = y^\nu R^a_{\ b\nu\mu} \mathrm{d}y^\mu. \tag{12}$$

A Taylor expansion of both sides of this equation at 0 then yields

$$\omega_{\mu b}^{a}(y) = \sum_{n=0}^{\infty} \frac{(y \cdot \partial_{x})^{n}}{n!(n+2)} y^{\nu} R^{a}_{b\nu\mu}(0) = \sum_{n=0}^{\infty} \frac{(y \cdot \nabla_{x})^{n}}{n!(n+2)} y^{\nu} R^{a}_{b\nu\mu}(0).$$
 (13)

This is, of course, nothing but the gauge theory identity eq. (3), specialized to the SO(D) case.

Next we act twice with \mathcal{L}_y on e. Using the absence of torsion, $de + \omega \wedge e = 0$, as well as the gauge condition eq. (11), one finds

$$\mathcal{L}_{v}e = \omega i_{v}e + \mathrm{d}i_{v}e, \tag{14}$$

$$\mathcal{L}_y \mathcal{L}_y e = (\mathcal{L}_y \omega) i_y e + \omega \mathcal{L}_y i_y e + \mathcal{L}_y di_y e.$$
 (15)

Using the gauge condition for the vielbein

$$i_y e^a = \delta^a_\mu y^\mu, \tag{16}$$

egs. (12), (14) and (15) can be combined to yield

$$(\mathcal{L}_y - 1)\mathcal{L}_y e^a = R^a_{\ \mu\nu d} y^\mu y^\nu e^d. \tag{17}$$

On the left hand side one can trivially rewrite

$$(\mathcal{L}_y - 1)\mathcal{L}_y e_\mu dy^\mu = \left[(y \cdot \partial)(y \cdot \partial + 1)e_\mu \right] dy^\mu.$$
 (18)

A Taylor expansion of both sides of eq. (17) at 0 then yields [15]

$$(y \cdot \nabla)^{k} e_{\mu}^{a}(0) = \frac{k-1}{k+1} (y \cdot \nabla)^{k-2} \left[R^{a}_{\alpha\beta b} y^{\alpha} y^{\beta} e_{\mu}^{b} \right] (0), \tag{19}$$

which expresses the Taylor coefficients of the vielbein in terms of the covariant derivatives of the Riemann tensor at 0.

Next we note that eq. (19) can be integrated to the following integral equation,

$$e^{a}_{\mu}(y) = \delta^{a}_{\mu} + y^{\alpha}y^{\beta} \int_{0}^{1} dv v (1 - v) R^{a}_{\alpha\beta b}(vy) e^{b}_{\mu}(vy).$$
 (20)

We decompose e(y) as

$$e(y) = \sum_{k=0}^{\infty} e_{(k)}(y)$$
 (21)

where k denotes the number of Riemann tensors appearing in a given term in the normal coordinate expansion of e. Obviously $e_{(k)}$ can be obtained by iterating k times eq. (20), and then replacing, under the integral, e^b_μ by $e^b_\mu(0) = \delta^b_\mu$. This yields

$$e_{(k)}(y) = \int_{0}^{1} dv_{1}(1-v_{1}) \dots \int_{0}^{1} dv_{k}(1-v_{k})v_{1}^{2k-1}v_{2}^{2k-3} \dots v_{k} \times \mathcal{R}(v_{1}y, y)\mathcal{R}(v_{1}v_{2}y, y) \dots \mathcal{R}(v_{1}v_{2}\dots v_{k}y, y).$$
(22)

Here we have introduced the abbreviation

$$\mathcal{R}^{a}{}_{b}(x,y) \equiv R^{a}{}_{\alpha\beta b}(x)y^{\alpha}y^{\beta}. \tag{23}$$

To arrive at the metric itself, we need also the transposed of eq. (22), which we can write as

$$e_{(k)}^{t}(y) = \int_{0}^{1} dv_{1}(1 - v_{1}) \dots \int_{0}^{1} dv_{k}(1 - v_{k})v_{1}v_{2}^{3} \dots v_{k}^{2k-1} \times \mathcal{R}(v_{1}v_{2} \dots v_{k}y, y)\mathcal{R}(v_{2} \dots v_{k}y, y) \dots \mathcal{R}(v_{k}y, y).$$
(24)

The final result for the metric $g = e^t e$ becomes

$$g(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e_{(i)}^{t}(y) e_{(j)}(y)$$

$$= \sum_{k=0}^{\infty} \int_{0}^{1} dv_{1}(1-v_{1}) \int_{0}^{1} dv_{2}(1-v_{2}) \dots \int_{0}^{1} dv_{k}(1-v_{k})$$

$$\times \sum_{l=0}^{k} v_{1} v_{2}^{3} \dots v_{l}^{2l-1} v_{l+1}^{2k-2l-1} v_{l+2}^{2k-2l-3} \dots v_{k}$$

$$\times \mathcal{R}(v_{1} v_{2} \dots v_{l} y, y) \mathcal{R}(v_{2} \dots v_{l} y, y) \dots \mathcal{R}(v_{l} y, y)$$

$$\times \mathcal{R}(v_{l+1} y, y) \mathcal{R}(v_{l+1} v_{l+2} y, y) \dots \mathcal{R}(v_{l+1} \dots v_{k} y, y). \tag{25}$$

This integral representation, which for a given value of k encodes the coefficients of all terms in the normal coordinate expansion having a fixed number of Riemann tensors, appears to be the closest possible analogue of the gauge theory formula eq. (2) 2 .

To explicitly obtain the coefficients, we now use eqs.(5),(11) again to covariantly Taylor expand all Riemann tensor factors,

$$\mathcal{R}(vy,y) = \sum_{\kappa=0}^{\infty} \frac{v^{\kappa} (y \cdot \nabla)^{\kappa}}{\kappa!} \mathcal{R}(0,y). \tag{26}$$

This leads to coefficient integrals which are easily calculated, with the result

$$e_{(k)}(y) = \sum_{\kappa_1,\dots,\kappa_k=0}^{\infty} \frac{C_{\kappa_1,\dots,\kappa_k}}{(\kappa_1 + \dots + \kappa_k + 2k + 1)!} (y \cdot \nabla)^{\kappa_1} \mathcal{R}(0,y) \dots (y \cdot \nabla)^{\kappa_k} \mathcal{R}(0,y)$$
(27)

where

$$C_{\kappa_1,\dots,\kappa_k} = \prod_{l=1}^k \binom{\kappa_l + \kappa_{l+1} + \dots + \kappa_k + 2k - 2l + 1}{\kappa_l}.$$
 (28)

Introducing the Pochammer symbol $(a)_n = a(a+1)\dots(a+n-1)$ this can also be written

$$e_{(k)}(y) = \sum_{\kappa_1, \dots, \kappa_k = 0}^{\infty} \prod_{l=1}^{k} \frac{(y \cdot \nabla)^{\kappa_l} \mathcal{R}(0, y)}{\kappa_l! (\kappa_l + \dots + \kappa_k + 2k - 2l + 2)_2}.$$
 (29)

Neither the integral formula eq. (25) nor the coefficient formula eq. (28) seem to have appeared in the literature before (though ref. [17] contains formulas equivalent to eq. (28)). In ref. [15] eq. (19) was instead used to derive a recursion formula for the normal coordinate expansion coefficients. Define the matrices [18]

$$\mathsf{E}_k = (e_\mu^a)_{,\mu_1\dots\mu_k}(0). \tag{30}$$

These are the k-th partial derivatives of the vielbein evaluated at the origin of the normal coordinate system. Define also

$$\mathsf{R}_{k} = R^{\mu \ \nu}_{(\mu_{1}\mu_{2} ; \dots \mu_{k})}(0) \tag{31}$$

²After submitting this paper we were informed by Dolgov and Khriplovich that they had published a similar result [16]. However, their eq. (35) concerns $x^{\mu}\partial_{\mu}g_{\alpha\beta}$. A solution for the metric (or the vielbein) itself is not given.

which one can consider as (symmetric) matrices in the indices μ and ν . Then we can rewrite eq. (19) as

$$(k+1)\mathsf{E}_k = (k-1)\mathsf{R}_k + \sum_{n=2}^{k-2} \binom{k-1}{n+1} \mathsf{R}_{k-n} (n+1)\mathsf{E}_n \tag{32}$$

for $k \geq 2$, with

$$E_0 = 1, \quad E_1 = 0.$$
 (33)

Here 1 denotes the D - dimensional unit matrix, and symmetrization on the k indices is understood for each summand. This recursion relation was used in [15] to list, in their eq. (2.6), the normal coordinate expansion through 8-th order (with several errors at 8-th order). Our eqs. (27), (28) resolve this recursion, as can be easily verified by rewriting them in the following form,

$$(k+1)\mathsf{E}_{k} = (k-1)\mathsf{R}_{k} + \sum_{\beta=1}^{\infty} \Big[\prod_{\alpha=1}^{\beta} \sum_{n_{\alpha}=2}^{n_{\alpha-1}-2} \binom{n_{\alpha-1}-1}{n_{\alpha}+1} \mathsf{R}_{n_{\alpha-1}-n_{\alpha}} \Big] (n_{\beta}-1)\mathsf{R}_{n_{\beta}}$$
(34)

(with $n_0 = k$).

To tenth order the coefficients are given explicitly in the appendix.

While the existence of a non-recursive formula for the normal coordinate coefficients may be of independent mathematical interest, we expect it also to become of practical use in computer-based high-order calculations of physically interesting quantities. In particular, in recent years rapid progress has been made in the calculation of heat-kernel coefficients and effective actions. Due to improvements in computerization and to the availability of new algorithms, this type of calculation can now be extended to orders which would have been unthinkable a few years ago [18, 19, 20]. This was also the original motivation for this work. Of course the uses of the normal coordinate expansion in physics are not restricted to quantum field theory; for example, our formulas may also be of relevance for the investigation of the validity of Huygen's principle in curved spaces (see, e.g., ref. [21] in [18]).

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Appendix: Coefficients to Tenth Order

To tenth order in the normal coordinate expansion we find for the vielbein e (writing $\{A\} = A + A^t$ and, par abus de language, R_k for $R_k y^k$)

$$\begin{split} e(y) &= \mathsf{I} + \tfrac{1}{3!}\mathsf{R}_2 + \tfrac{1}{4!}2\mathsf{R}_3 + \tfrac{1}{5!}(3\mathsf{R}_4 + \mathsf{R}_2^2) + \tfrac{1}{6!}(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3) \\ &+ \tfrac{1}{7!}(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)) \\ &+ \tfrac{1}{8!}(6\mathsf{R}_7 + 20\mathsf{R}_5\mathsf{R}_2 + 30\mathsf{R}_4\mathsf{R}_3 + 6\mathsf{R}_3(3\mathsf{R}_4 + \mathsf{R}_2^2) + \mathsf{R}_2(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3)) \\ &+ \tfrac{1}{9!}(7\mathsf{R}_8 + 35\mathsf{R}_6\mathsf{R}_2 + 70\mathsf{R}_5\mathsf{R}_3 + 21\mathsf{R}_4(3\mathsf{R}_4 + \mathsf{R}_2^2) + 7\mathsf{R}_3(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)) \\ &+ \tfrac{1}{10!}(8\mathsf{R}_9 + 56\mathsf{R}_7\mathsf{R}_2 + 140\mathsf{R}_6\mathsf{R}_3 + 56\mathsf{R}_5(3\mathsf{R}_4 + \mathsf{R}_2^2) + 28\mathsf{R}_4(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3) \\ &+ 8\mathsf{R}_3(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)) \\ &+ \mathsf{R}_2(6\mathsf{R}_7 + 20\mathsf{R}_5\mathsf{R}_2 + 30\mathsf{R}_4\mathsf{R}_3 + 6\mathsf{R}_3(3\mathsf{R}_4 + \mathsf{R}_2^2) + \mathsf{R}_2(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3)) \\ &+ \tfrac{1}{11!}(9\mathsf{R}_{10} + 84\mathsf{R}_8\mathsf{R}_2 + 252\mathsf{R}_7\mathsf{R}_3 + 126\mathsf{R}_6(3\mathsf{R}_4 + \mathsf{R}_2^2) \\ &+ 84\mathsf{R}_5(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3) + 36\mathsf{R}_4(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)) \\ &+ 9\mathsf{R}_3(6\mathsf{R}_7 + 20\mathsf{R}_5\mathsf{R}_2 + 30\mathsf{R}_4\mathsf{R}_3 + 6\mathsf{R}_3(3\mathsf{R}_4 + \mathsf{R}_2^2) + \mathsf{R}_2(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3)) \\ &+ \mathsf{R}_2(7\mathsf{R}_8 + 35\mathsf{R}_6\mathsf{R}_2 + 70\mathsf{R}_5\mathsf{R}_3 + 21\mathsf{R}_4(3\mathsf{R}_4 + \mathsf{R}_2^2) + 7\mathsf{R}_3(4\mathsf{R}_5 + 4\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_2\mathsf{R}_3)) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3^2 + \mathsf{R}_2(3\mathsf{R}_4 + \mathsf{R}_2^2)))) \\ &+ \mathsf{R}_2(5\mathsf{R}_6 + 10\mathsf{R}_4\mathsf{R}_2 + 10\mathsf{R}_3$$

and for the metric $g = e^t e$

$$\begin{split} g(y) &= \mathsf{I} + \tfrac{1}{2!} \tfrac{2}{3} \mathsf{R}_2 + \tfrac{1}{3!} \, \mathsf{R}_3 + \tfrac{1}{4!} \tfrac{6}{5} (\mathsf{R}_4 + \tfrac{8}{9} \mathsf{R}_2^2) + \tfrac{1}{5!} \tfrac{4}{3!} (\mathsf{R}_5 + 2 \{ \mathsf{R}_2 \mathsf{R}_3 \}) \\ &+ \tfrac{1}{6!} \tfrac{17}{7} (\mathsf{R}_6 + \tfrac{17}{5} \{ \mathsf{R}_2 \mathsf{R}_4 \} + \tfrac{11}{2!} \mathsf{R}_3^2 + \tfrac{8}{5} \mathsf{R}_2^3) \\ &+ \tfrac{1}{7!} \tfrac{3}{2} (\mathsf{R}_7 + \tfrac{46}{9} \{ \mathsf{R}_2 \mathsf{R}_5 \} + 11 \{ \mathsf{R}_3 \mathsf{R}_4 \} + \tfrac{62}{9} \mathsf{R}_2 \mathsf{R}_3 \mathsf{R}_2 + \tfrac{41}{9} \{ \mathsf{R}_2^2 \mathsf{R}_3 \}) \\ &+ \tfrac{1}{8!} \tfrac{14}{9} (\mathsf{R}_8 + \tfrac{50}{7} \{ \mathsf{R}_2 \mathsf{R}_6 \} + 19 \{ \mathsf{R}_3 \mathsf{R}_5 \} + \tfrac{126}{5} \mathsf{R}_4^2 + \tfrac{130}{7} \mathsf{R}_2 \mathsf{R}_4 \mathsf{R}_2 \\ &+ \tfrac{339}{35} \{ \mathsf{R}_2^2 \mathsf{R}_4 \} + 14 \mathsf{R}_3 \mathsf{R}_2 \mathsf{R}_3 + \tfrac{163}{7} \{ \mathsf{R}_2 \mathsf{R}_3 \mathsf{R}_3 \} + \tfrac{128}{35} \mathsf{R}_2^4) \\ &+ \tfrac{1}{9!} \tfrac{8}{5} (\mathsf{R}_9 + \tfrac{19}{2} \{ \mathsf{R}_2 \mathsf{R}_7 \} + 30 \{ \mathsf{R}_3 \mathsf{R}_6 \} + 49 \{ \mathsf{R}_4 \mathsf{R}_5 \} + 40 \mathsf{R}_2 \mathsf{R}_5 \mathsf{R}_2 + 18 \{ \mathsf{R}_2^2 \mathsf{R}_5 \} + 85 \mathsf{R}_3^3 \\ &+ \tfrac{113}{2!} \{ \mathsf{R}_2 \mathsf{R}_3 \mathsf{R}_4 \} + 32 \{ \mathsf{R}_3 \mathsf{R}_2 \mathsf{R}_4 \} + \tfrac{145}{2} \{ \mathsf{R}_2 \mathsf{R}_4 \mathsf{R}_3 \} + \tfrac{23}{2} \{ \mathsf{R}_2^3 \mathsf{R}_3 \} + \tfrac{41}{2!} \{ \mathsf{R}_2^2 \mathsf{R}_3 \mathsf{R}_2 \}) \\ &+ \tfrac{1}{10!} \tfrac{18}{11} (\mathsf{R}_{10} + \tfrac{329}{27} \{ \mathsf{R}_2 \mathsf{R}_8 \} + \tfrac{89}{2} \{ \mathsf{R}_3 \mathsf{R}_7 \} + 86 \{ \mathsf{R}_4 \mathsf{R}_6 \} + \tfrac{952}{9} \mathsf{R}_5^2 + \tfrac{2030}{27} \mathsf{R}_2 \mathsf{R}_6 \mathsf{R}_2 \\ &+ \tfrac{829}{27} \{ \mathsf{R}_2^2 \mathsf{R}_6 \} + 78 \mathsf{R}_4 \mathsf{R}_2 \mathsf{R}_4 + \tfrac{593}{3} \{ \mathsf{R}_4 \mathsf{R}_4 \mathsf{R}_2 \} + 305 \mathsf{R}_3 \mathsf{R}_4 \mathsf{R}_3 + \tfrac{443}{2} \{ \mathsf{R}_3^2 \mathsf{R}_4 \} \\ &+ \tfrac{3164}{27} \{ \mathsf{R}_2 \mathsf{R}_3 \mathsf{R}_5 \} + \tfrac{575}{9} \{ \mathsf{R}_3 \mathsf{R}_2 \mathsf{R}_5 \} + \tfrac{4775}{27} \{ \mathsf{R}_2 \mathsf{R}_5 \mathsf{R}_3 \} + \tfrac{245}{9} \{ \mathsf{R}_2^3 \mathsf{R}_4 \} + \tfrac{1889}{27} \{ \mathsf{R}_2^2 \mathsf{R}_4 \mathsf{R}_2 \} \\ &+ \tfrac{4207}{54} \{ \mathsf{R}_2^2 \mathsf{R}_3^2 \} + \tfrac{1879}{27} \{ \mathsf{R}_2 \mathsf{R}_3 \mathsf{R}_2 \mathsf{R}_3 \} + \tfrac{3472}{27} (\mathsf{R}_2 \mathsf{R}_3^2 \mathsf{R}_2 + \mathsf{R}_3 \mathsf{R}_2^2 \mathsf{R}_3) + \tfrac{256}{27} \mathsf{R}_2^5) \\ &+ O(11). \end{split}$$

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